Overview of Gyrofluid Equations and their Numerical Solution

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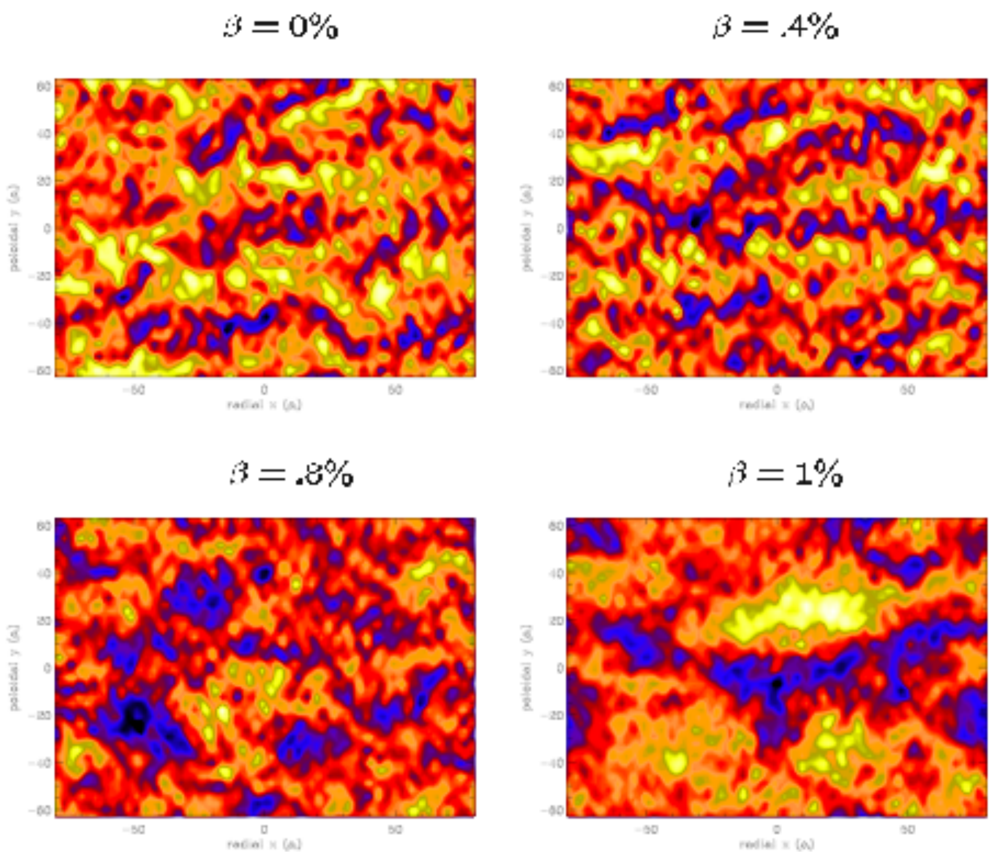

BOUT++ Workshop
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GENERAL ATOMICS

Definition of Gyrofluid

- **"Landau Fluid" equations are fluid moment equations which are truncated with closure terms incorporating kinetic effects**
  - Landau damping (linear), Toroidal drift resonance…
- **"Gyro-Landau Fluid" or "Gyrofluid" equations are fluid moments of the 5D gyrokinetic equation, closed so as to maintain FLR and kinetic effects**
  - Landau damping, linear and nonlinear FLR, toroidal drifts and drift resonance, trapping
  - Able to accurately reproduce linear GK physics, and provide reasonable agreement with nonlinear GK simulations, while being much more efficient
  - Eg: GRYFFIN (Beer, Dorland, Hammett, Snyder) and Waltz GLF simulation codes, GLF23 and TGLF linear models

Vlasov, Boltzmann, Liouville Eq:

Particle Distribution

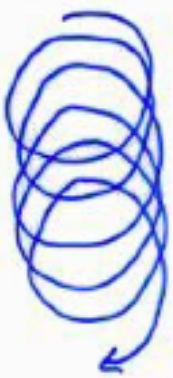
$$\frac{\partial f(\underset{\sim}{x}, \underset{\sim}{v}, t)}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \frac{q}{m} \underbrace{\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} \right) \cdot \frac{\partial f}{\partial \underset{\sim}{v}}} = C(f)$$

Nonlinear, $\underset{\sim}{E}$ & $\underset{\sim}{B}$ depend on f through Maxwell's Eqs.

Nonlinear Gyrokinetic Eq. 1982–88

(Frieman & Chen, W.W. Lee, Dubin, Krommes, Hahm, Brizard...)

linear gyrokinetics 1960's & 70's.



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+ ↓ $\underset{\sim}{V}_{drift}$

$$\frac{c}{B^2} \langle \underset{\sim}{E} \rangle \times \underset{\sim}{B} + \cdots$$

Possible to eliminate fast gyrofrequency time scales & retain nonlinear dynamics

& $k_\perp \rho_i \sim 1$

Comparison of Gyrofluid and Traditional MHD-like 2-fluid Equations

- **GF: Moments of 5D GK eqn rather than 6D K eqn**
  - Fast timescales and short scale lenaths eliminated before moments are taken $\frac{\omega}{\Omega_i} \sim \frac{k_\parallel v_{ti}}{\Omega_i} \sim \frac{e\phi}{T} \sim \frac{\delta B}{B} \sim \frac{F_1}{F_0} \sim \frac{\rho_i}{L} \sim \varepsilon \ll 1, \qquad k_\perp \rho_i \sim 1,$
- **GF: Moments taken in gyro-center space**
  - Gyro-viscous cancellation natural, algebra easier
- **GF: Closures derived by matching to linear kinetic response, rather than high collisionality**
  - Can accurately reproduce kinetic physics in both low and high collisionality limits
- **GF: FLR and closure terms take a form which can be efficiently evaluated *in k-space***
  - Much more challenging to evaluate in x-space

Deriving Gyrofluid Equations

- **Start with GK eqn:**

$$\frac{\partial F}{\partial t} + (v_\parallel \tilde{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla F + \left[\frac{e}{m}\tilde{E}_\parallel - \mu\tilde{\mathbf{b}}\cdot\nabla B + v_\parallel(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right]\frac{\partial F}{\partial v_\parallel} = C(F).$$

$$\frac{\omega}{\Omega_i} \sim \frac{k_\parallel v_{ti}}{\Omega_i} \sim \frac{e\phi}{T} \sim \frac{\delta B}{B} \sim \frac{F_1}{F_0} \sim \frac{\rho_i}{L} \sim \varepsilon \ll 1, \qquad k_\perp \rho_i \sim 1,$$

- **Put it in conservative form:**

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{f}B &+ B\nabla_\parallel \tilde{f}v_\parallel + \mathbf{v}_\phi\cdot\nabla[(F_0+\tilde{f})BJ_0] - \mathbf{v}_{A_\parallel}\cdot\nabla[(F_0+\tilde{f})B\frac{v_\parallel}{c}J_0] \\
&+ 2F_0BJ_0 i\omega_d\frac{e\phi}{T} - F_0B\frac{v_\parallel}{c}J_0 i\omega_d\frac{eA_\parallel}{T} + F_0BJ_1\frac{\alpha}{2}i\omega_d\frac{e\phi}{T} \\
&- F_0B\frac{v_\parallel}{c}J_1\frac{\alpha}{2}i\omega_d\frac{eA_\parallel}{T} + \frac{i\omega_d}{v_t^2}[\tilde{f}B(v_\parallel^2+\mu B)] - \frac{e}{mc}\frac{\partial F_0}{\partial v_\parallel}BJ_0\frac{\partial A_\parallel}{\partial t} \\
&- \frac{e}{m}\nabla_\parallel(\frac{\partial F_0}{\partial v_\parallel}BJ_0\phi) + \frac{e}{m}J_0\phi\frac{\partial F_0}{\partial v_\parallel}B(1-\frac{\mu B}{v_t^2})\nabla_\parallel \ln B \\
&+ \frac{e}{m}\frac{\partial F_0}{\partial v_\parallel}J_{0A_\parallel}J_{0\phi}(\hat{\mathbf{b}}\times\nabla A_\parallel)\cdot\nabla\phi - \mu B^2\frac{\partial\tilde{f}}{\partial v_\parallel}\nabla_\parallel \ln B \\
&- \frac{\partial F_0}{\partial v_\parallel}BJ_0\frac{\mu B}{c}i\omega_d\frac{eA_\parallel}{T} - \frac{\partial}{\partial v_\parallel}(F_0BJ_0v_\parallel)i\omega_d\frac{e\phi}{T} = 0.
\end{aligned}$$

- **Take moments:**

Velocity space moments are often defined in terms of the total distribution function F. Here we again separate F into equilibrium and fluctuating components $F = F_0 + \tilde{f}$.[6] Velocity space moments of

$$F_0 = F_M = \frac{n_0}{(2\pi v_t^2)^{3/2}}e^{-v_\parallel^2/2v_t^2-\mu B/v_t^2}$$

are all well defined. We define the following moments of the fluctuating distribution:

$$\begin{aligned}
\tilde{n} &= \int \tilde{f}\,d^3v & n_0\tilde{u}_\parallel &= \int \tilde{f}v_\parallel\,d^3v \\
\tilde{p}_\parallel &= m\int \tilde{f}v_\parallel^2\,d^3v & \tilde{p}_\perp &= m\int \tilde{f}B\mu\,d^3v \\
\tilde{q}_\parallel &= -3mv_t^2n_0\tilde{u}_\parallel + m\int \tilde{f}v_\parallel^3\,d^3v & \tilde{q}_\perp &= -mv_t^2n_0\tilde{u}_\parallel + m\int \tilde{f}B\mu\, v_\parallel\,d^3v \\
\tilde{r}_{\parallel,\parallel} &= m\int \tilde{f}v_\parallel^4\,d^3v & \tilde{r}_{\parallel,\perp} &= m\int \tilde{f}B\mu\, v_\parallel^2\,d^3v \\
\tilde{r}_{\perp,\perp} &= m\int \tilde{f}B^2\mu^2\,d^3v & \tilde{s}_{\perp,\perp} &= -2mv_t^4n_0\tilde{u}_\parallel + m\int \tilde{f}B^2\mu^2v_\parallel\,d^3v \\
\tilde{s}_{\parallel,\parallel} &= -15mv_t^4n_0\tilde{u}_\parallel + m\int \tilde{f}v_\parallel^5\,d^3v & \tilde{s}_{\parallel,\perp} &= -3mv_t^4n_0\tilde{u}_\parallel + m\int \tilde{f}B\mu\, v_\parallel^3\,d^3v,
\end{aligned}$$

Closure of high moments (3+1 or 4+2) preserves particle, momentum and energy conservation

Deriving Closure Terms

- **Closures needed for FLR, parallel, toroidal drift and mirror terms**
- **Landau damping:**

Nothing is really damped in Landau damping.
Plase mixing moves fluctuations to fine scales in v space.
Once at small scales we assume they are damped by collisions. Good assumption for turbulence, not so good for special cases like plasma echo

As an illustration, consider the one dimensional kinetic equation

$$\frac{\partial f}{\partial t} + v_{\parallel}\frac{\partial f}{\partial z} = \delta(t) f_0(z, v), \tag{3.74}$$

where f_0 provides the initial condition. The solution to this simple equation $f(z,v,t) = f_0(z - vt, v)H(t)$, provides Green's function which can be used to solve more general problems with additional source terms, such as the electric field $-(e/m)E_{\parallel}\frac{\partial F_M}{\partial v}$. Consider an initial condition with a small single harmonic perturbation $f_0 = (n_0 + n_1 e^{ikz})F_M(v)$. The general solution is just $(n_0 + n_1 e^{ik(z-vt)})$, which simply oscillates in time at $\omega = kv$ and does not damp. However, upon taking velocity space moments, the velocity integration introduces mixing of the phases as follows:

$$n(z,t) = \int f\,dv = n_0 + n_1 \frac{e^{ikz}}{\sqrt{2\pi v_t^2}} \int dv\, e^{-ikvt} e^{-v^2/(2v_t^2)}. \tag{3.75}$$

The perturbed density $n_1 = n_{1(t=0)} e^{-k^2 v_t^2 t^2/2}$ decays with a Gaussian time dependence. This decay due to linear Landau damping is not captured by a simple fluid model with a finite number of moments, and hence it must be accounted for in the fluid closure if it is to be included in a fluid model.

Deriving Closure Terms: Practice

- **Kinetic linear response:**

$$n_{1s} = -\frac{in_0}{k_z T_{\parallel 0s}} e_s E_\parallel \mathcal{R}(\zeta_s) + \frac{B_1 n_0}{B_0}\left[1 - \frac{T_{\perp 0s}}{T_{\parallel 0s}}\mathcal{R}(\zeta_s)\right]$$

where $\zeta_s = \omega/\sqrt{2}|k_z|v_{t\parallel_s}$ is the normalized frequency, and $\mathcal{R}(\zeta_s) = 1 + \zeta_s Z(\zeta_s)$

- **GF closure form:**
- **D's determined by matching K response in small and large z limit**

$$r_{\parallel,\parallel_s} = 3v_{t\parallel_s}^2(2p_{\parallel_s} - T_{\parallel 0s}n) + c_\parallel n_0 v_{t\parallel_s}^2 T_{\parallel_s} - \sqrt{2}D_\parallel v_{t\parallel_s}\frac{ik_\parallel q_{\parallel_s}}{|k_\parallel|}$$

$$r_{\parallel,\perp_s} = v_{t\perp_s}^2 p_{\parallel_s} + v_{t\parallel_s}^2 p_{\perp s} - v_{t\parallel_s}^2 T_{\perp 0s}n - \sqrt{2}D_\perp v_{t\parallel_s}\frac{ik_\parallel q_{\perp s}}{|k_\parallel|}$$

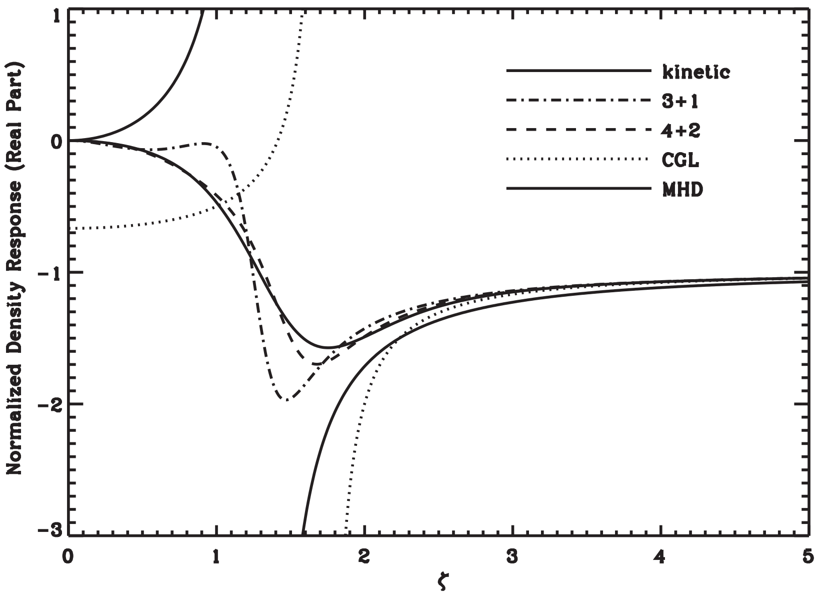
The density response is then:

$$n_{1s} = -\frac{in_0}{k_z T_{\parallel 0s}} e_s E_\parallel \mathcal{R}_4(\zeta_s) + \frac{B_1 n_0}{B_0}\left[1 - \frac{T_{\perp 0s}}{T_{\parallel 0s}}\mathcal{R}_4(\zeta_s)\right] \qquad \text{(C.36)}$$

where $\mathcal{R}_4(\zeta_s)$ is a four-pole model of the electrostatic response function $\mathcal{R}(\zeta_s)$:

$$\mathcal{R}_4(\zeta_s) = \frac{4 - 2i\sqrt{\pi}\zeta_s + (8-3\pi)\zeta_s^2}{4 - 6i\sqrt{\pi}\zeta_s + (16-9\pi)\zeta_s^2 + 4i\sqrt{\pi}\zeta_s^3 + (6\pi-16)\zeta_s^4}. \qquad \text{(C.37)}$$

Accurately reproduces kinetic response and linear growth rates



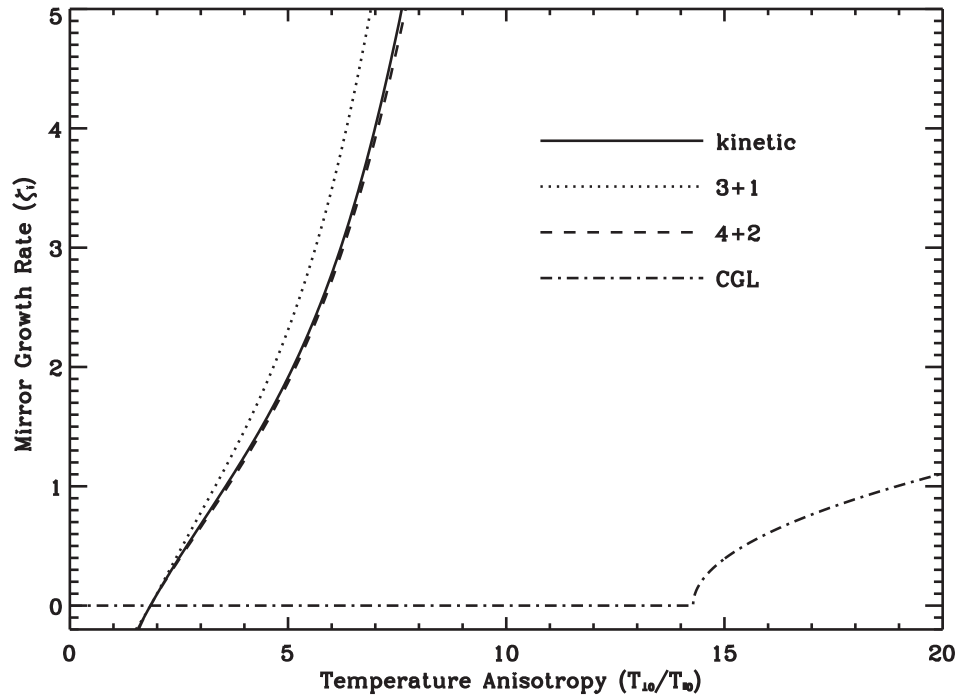


Figure C.3: Linear growth rate of the mirror instability ($k^2 \gg k_\parallel^2$) as predicted by kinetic theory, 3+1 and 4+2 Landau MHD models, and CGL theory (ideal MHD cannot predict the mirror growth rate as it posits an isotropic pressure). The normalized growth rate ($\zeta_i = Im(\omega)/\sqrt{2}|k_\parallel|v_{T\parallel_i}$) is plotted versus the temperature anisotropy ($T_{\perp 0}/T_{\parallel 0}$) at constant $\beta = \{(2/3)p_{\perp 0} + (1/3)p_{\parallel 0}\}/(B_0^2/8\pi)$. The parameters chosen are $Z = 1$, $T_{\perp 0i} = T_{\perp 0e}$, $T_{\parallel 0i} = T_{\parallel 0e}$, $\beta = 1$ and $\sqrt{m_i/m_e} = 40$.

GENERAL ATOMICS

Adding FLR, Toroidal Drift & Mirror Closures gives final EM Gyrofluid equations

Toroidal Ion Gyrofluid Equations

$$
\begin{aligned}
\frac{dn}{dt} &+ [\tfrac{1}{2}\hat{\nabla}^2 \mathbf{v}_\Psi]\cdot\nabla T_\perp - [\tfrac{1}{2}\hat{\nabla}^2 \mathbf{v}_A]\cdot\nabla q_\perp + B\hat{\nabla}_\parallel \frac{u_\parallel}{B} - \left(1+\frac{\eta_\perp}{2}\nabla_\perp^2\right) i\omega_* \Psi \\
&+ \left(2+\frac{1}{2}\hat{\nabla}_\perp^2\right) i\omega_d \Psi + i\omega_d (p_\parallel + p_\perp) = 0 \\
\frac{du_\parallel}{dt} &+ [\tfrac{1}{2}\hat{\nabla}^2 \mathbf{v}_\Psi]\cdot\nabla q_\perp + B\hat{\nabla}_\parallel \frac{p_\parallel}{B} + \frac{\partial A_\parallel}{\partial t} + \hat{\nabla}_\parallel \Psi + \left(1+\eta_\parallel+\frac{\eta_\perp}{2}\hat{\nabla}^2\right) i\omega_* A_\parallel \\
&- [\tfrac{1}{2}\hat{\nabla}^2 \mathbf{v}_A]\cdot\nabla T_\perp + \left(p_\perp + \frac{1}{2}\hat{\nabla}_\perp^2 \Psi\right)\nabla_\parallel \ln B + i\omega_d (q_\parallel + q_\perp + 4u_\parallel) = 0 \\
\frac{dp_\parallel}{dt} &+ [\tfrac{1}{2}\hat{\nabla}^2 \mathbf{v}_\Psi]\cdot\nabla T_\parallel + B\hat{\nabla}_\parallel \frac{q_\parallel + 3u_\parallel}{B} + 2(q_\perp + u_\parallel)\nabla_\parallel \ln B \\
&- \left(1+\eta_\parallel+\frac{\eta_\perp}{2}\hat{\nabla}_\perp^2\right) i\omega_* \Psi + \left(4+\frac{1}{2}\hat{\nabla}^2\right) i\omega_d \Psi + i\omega_d (7p_\parallel + p_\perp - 4n) \\
&+ 2|\omega_d|(\nu_1 T_\parallel + \nu_2 T_\perp) = -\frac{2}{3}\nu_{ii}(p_\parallel - p_\perp) \\
\frac{dp_\perp}{dt} &+ [\tfrac{1}{2}\hat{\nabla}^2 \mathbf{v}_\Psi]\cdot\nabla p_\perp + [\hat{\nabla}_\perp^2 \mathbf{v}_\Psi]\cdot\nabla T_\perp - [\tfrac{1}{2}\hat{\nabla}_\perp^2 \mathbf{v}_A]\cdot\nabla (q_\perp + u_\parallel) \\
&+ B^2\hat{\nabla}_\parallel \frac{q_\perp + u_\parallel}{B^2} - \left[1+\frac{1}{2}\hat{\nabla}_\perp^2 + \eta_\perp\left(1+\frac{1}{2}\hat{\nabla}^2 + \hat{\nabla}^2\right)\right] i\omega_* \Psi \\
&+ \left(3+2\hat{\nabla}^2+3\hat{\nabla}^2\right) i\omega_d \Psi + \left(3+\frac{3}{2}\hat{\nabla}_\perp^2 + \hat{\nabla}_\perp^2\right) i\omega_d \Psi \\
&+ i\omega_d (5p_\perp + p_\parallel - 3n) + 2|\omega_d|(\nu_3 T_\parallel + \nu_4 T_\perp) = \frac{1}{3}\nu_{ii}(p_\parallel - p_\perp) \\
\frac{dq_\parallel}{dt} &+ 3\hat{\nabla}_\parallel T_\parallel + \alpha_\parallel \nabla_\parallel T_\parallel + \sqrt{2}D_\parallel |k_\parallel| q_\parallel + i\omega_d (-3q_\parallel - 3q_\perp + 6u_\parallel) \\
&+ 3\eta_\parallel i\omega_* A_\parallel + |\omega_d|(\nu_5 u_\parallel + \nu_6 q_\parallel + \nu_7 q_\perp) = -\nu_{ii} q_\parallel \\
\frac{dq_\perp}{dt} &+ [\tfrac{1}{2}\hat{\nabla}^2 \mathbf{v}_\Psi]\cdot\nabla u_\parallel + [\hat{\nabla}_\perp^2 \mathbf{v}_\Psi]\cdot\nabla q_\perp - [\hat{\nabla}_\perp^2 - \tfrac{1}{2}\hat{\nabla}_\perp^2]\mathbf{v}_A\cdot\nabla T_\perp + \hat{\nabla}_\parallel T_\perp \\
&+ \left[\eta_\perp(1+\hat{\nabla}^2) + (1+\eta_\parallel)\frac{1}{2}\hat{\nabla}^2\right] i\omega_* A_\parallel + \frac{1}{2}\hat{\nabla}_\perp^2\left(\frac{dA_\parallel}{dt} + \hat{\nabla}_\parallel \Psi - i\omega_d A_\parallel\right) \\
&+ \sqrt{2}D_\perp |k_\parallel| q_\perp + \left(p_\perp - p_\parallel + \hat{\nabla}^2\Psi - \frac{1}{2}\hat{\nabla}_\perp^2 \Psi\right)\nabla_\parallel \ln B \\
&+ i\omega_d (-q_\parallel - q_\perp + u_\parallel) + |\omega_d|(\nu_8 u_\parallel + \nu_9 q_\parallel + \nu_{10} q_\perp) = -\nu_{ii} q_\perp
\end{aligned}
$$

where $\hat{\nabla}_\parallel = \nabla_\parallel - \hat{\mathbf{b}}\times\nabla A_\parallel\cdot\nabla$, $\mathbf{v}_A = \hat{\mathbf{b}}\times\nabla A_\parallel$, $A_\parallel = \Gamma_0^{1/2}A_\parallel$.

$$n_e = \bar{n}_i - (1-\Gamma_0)\phi,$$

$$\nabla_\perp^2 A_\parallel = -\frac{\tau\beta_e}{2}(\bar{u}_{\parallel i} - u_{\parallel e}),$$

Can simplify for special cases,

eg low mass electrons:

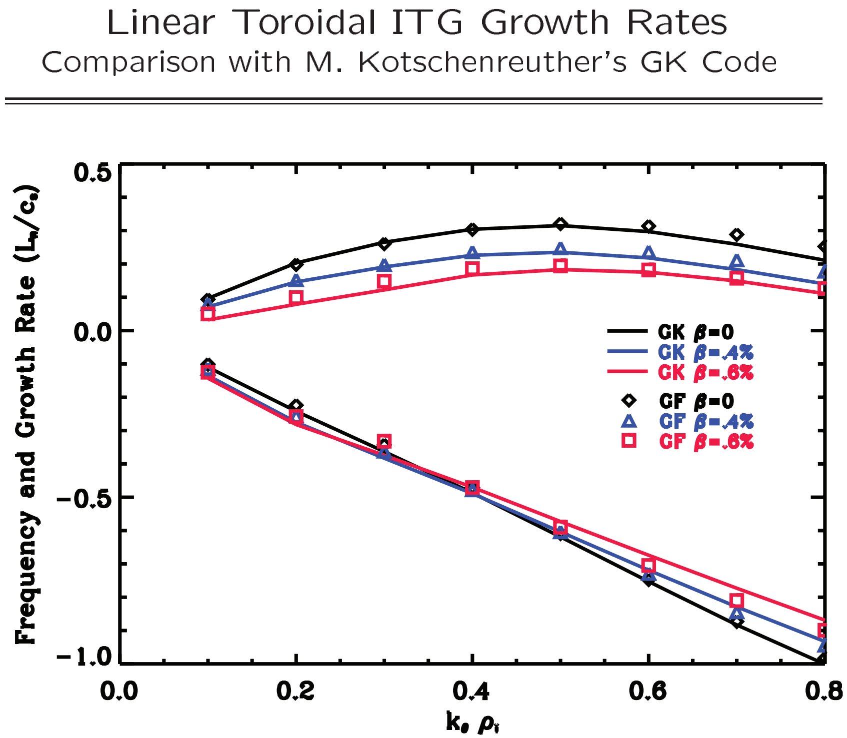
$$\frac{\partial n_e}{\partial t} + \mathbf{v}_E\cdot\nabla n_e + B\bar{\nabla}_\parallel \frac{u_{\parallel e}}{B} - i\omega_*\phi + 2i\omega_d\left(\phi - \frac{n_e}{\tau} - T_e\right) = 0,$$

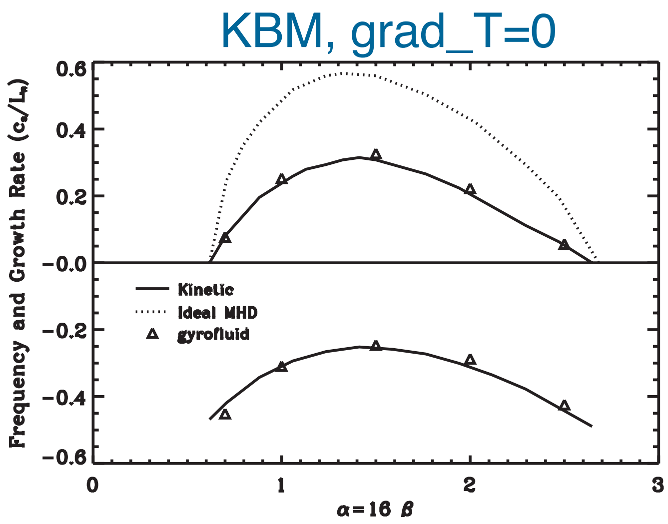
$$\frac{\partial A_\parallel}{\partial t} + \bar{\nabla}_\parallel\phi - \frac{1}{\tau}\bar{\nabla}_\parallel n_e - \frac{1}{\tau}i\omega_* A_\parallel - \sqrt{\frac{\pi}{2\tau}\frac{m_e}{m_i}}|k_\parallel| u_{\parallel e} = \nu_{ei}\frac{m_e}{m_i}(u_{\parallel e} - u_{\parallel i}),$$

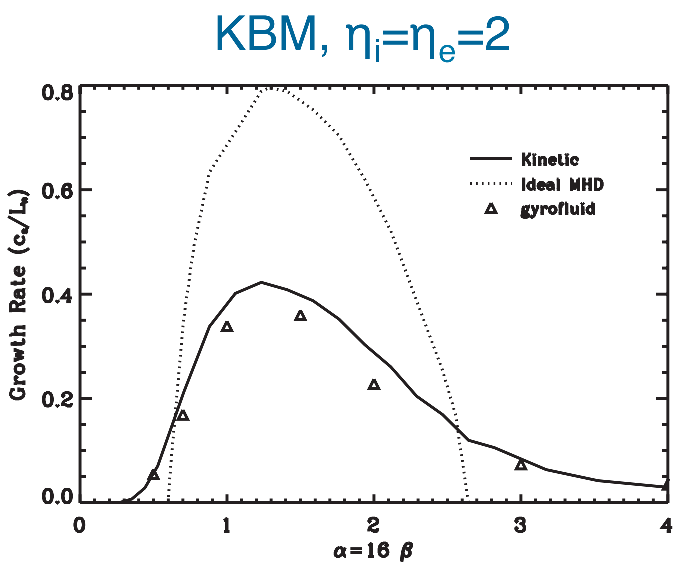
$$\bar{\nabla}_\parallel T_e = -\frac{\eta_e}{\tau}i\omega_* A_\parallel.$$

GENERAL ATOMICS

Adding FLR, Toroidal Drift & Mirror Closures allows accurate treatment of GK drift modes

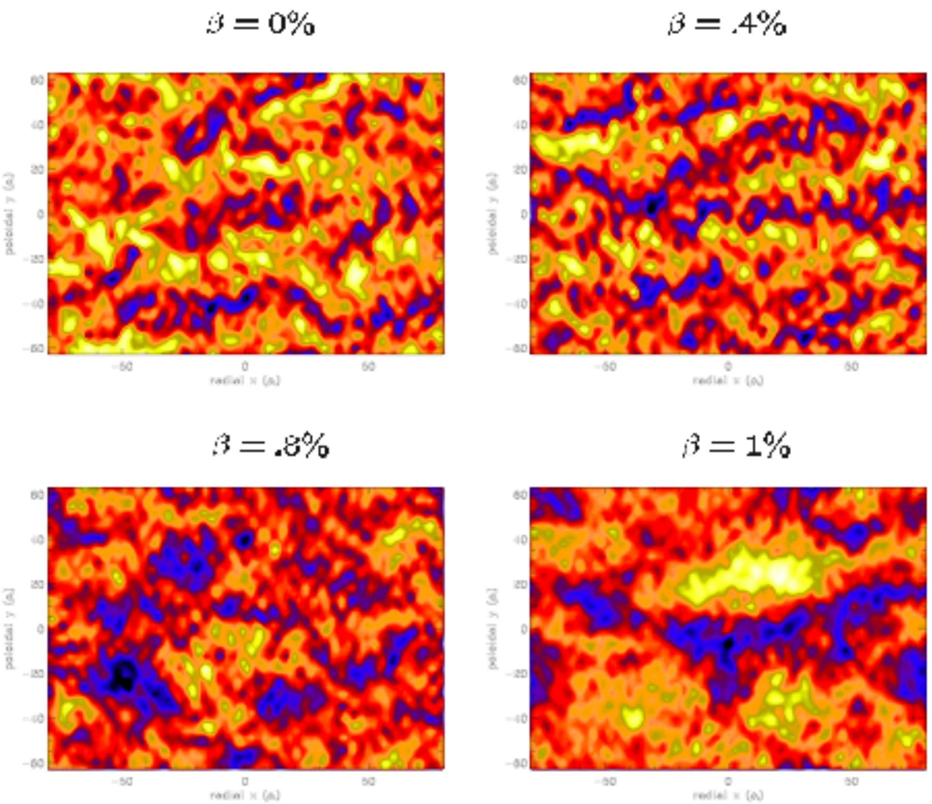


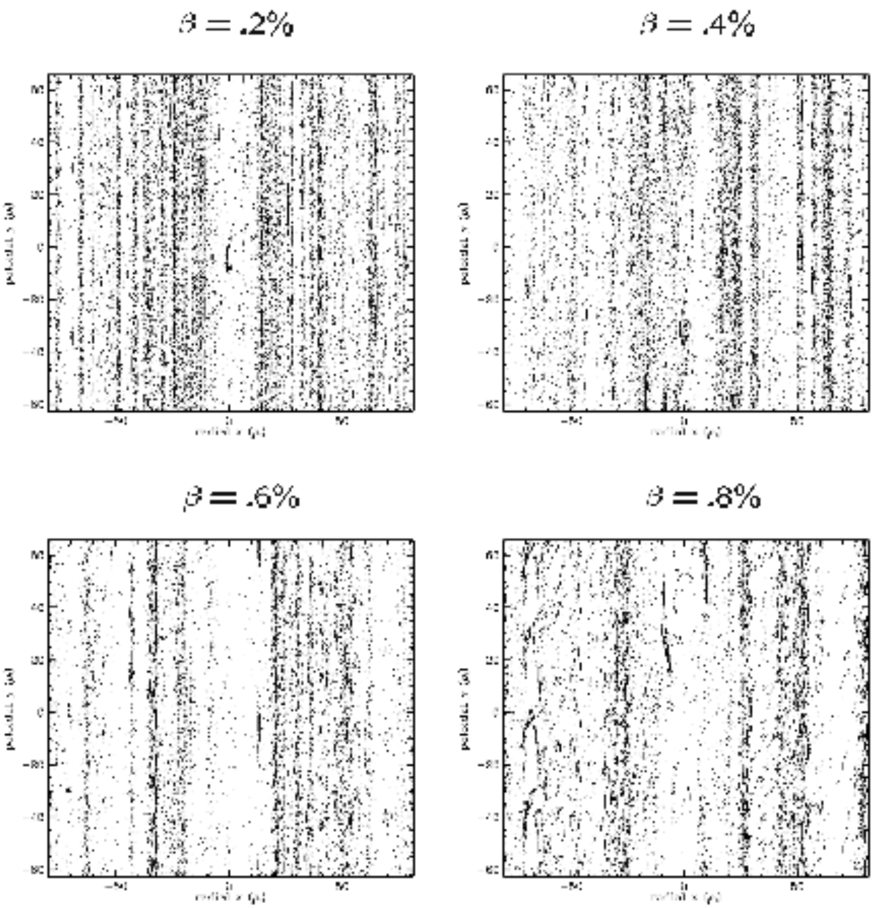




- **Reproduces Growth Rate for finite-β ITG as well as kinetic ballooning mode instability**
  - KBM unstable below ideal threshold when temperature gradient is finite

GF Equations used in NL flux tube simulations of EM drift modes





- **Small reduction in flux at low beta, then increase as KBM threshold approached**
  - Character of turbulence changes, B field stochastic, near KBM limit

Comparison of Methods for Solving GK Eqn

- **GK Eulerian or Continuum methods (eg GS2, GYRO, GENE)**
  - Grid v space and directly solve 5D equation
  - Can choose v space coords and grid for efficiency
  - Collisions straightforward to implement in principle, no noise issue
- **Particle-in-Cell (eg GTC/GTS, Parker's GEM…)**
  - Use markers (sometimes called superparticles or particles) to resolve velocity space in Monte-Carlo-like fashion
  - Relatively straightforward to code and scale, can get good v-space resolution via time averaging
  - Noise issues, and challenging to implement realistic collisions
- **Gyrofluid**
  - Take ~6 moments of GK eqn, kinetic closures, conservative
  - Moments are v-space grid, ~10-100 times more efficient
  - Nonlinear kinetic damping not treated
  - Some closures artificially damp R-H zonal flow, correctable problem
    - Myth: this is source of IFS-PPPL controversy; Reality: relatively small effect

Issues for GF in edge turbulence and ELM problems

- **Small perturbations assumed in closure derivation**
- **Need extension to full B, kink term**
- **Much of GF efficiency comes from flux tube, k-space**
  - Closure terms become integral operators in real space
    - Efficient evaluation is challenging
    - Simplification (eg localization or Scott constant method)

$$q_{\parallel s} = -8 n_0 v_{t\parallel s}^2 \frac{i k_\parallel T_{\parallel s}}{(\sqrt{8\pi}|k_\parallel| v_{t\parallel s} + (3\pi - 8)\nu_s)}$$

$$q_{\perp s} = -\frac{n_0 v_{t\parallel s}^2 i k_\parallel T_{\perp s}}{(\sqrt{\frac{\pi}{2}}|k_\parallel| v_{t\parallel s} + \nu_s)} + \left(1 - \frac{T_{\perp 0s}}{T_{\parallel 0s}}\right) \frac{n_0 v_{t\parallel s}^2 T_{\perp 0s} i k_\parallel B_1}{B_0(\sqrt{\frac{\pi}{2}}|k_\parallel| v_{t\parallel s} + \nu_s)}$$

$$q_{\parallel s}(z) = -n_0 \left(\frac{2}{\pi}\right)^{\frac{3}{2}} v_{t\parallel s} \int_0^\infty dz' \frac{T_{\parallel s}(z+z') - T_{\parallel s}(z-z')}{z'},$$

$$q_{\parallel s} = -n_0 \left(\frac{2}{\pi}\right)^{\frac{3}{2}} v_{t\parallel s} \int_0^\infty d\hat{z}' g(\hat{z}') \left[T_{\parallel s}(\hat{z}+\hat{z}') - T_{\parallel s}(\hat{z}-\hat{z}')\right]$$

$$g(\hat{z}) = \int_0^\infty d\hat{k} \frac{\hat{k}}{\hat{k}+1} \sin(\hat{k}\hat{z})$$

Discussion

- **Principal advantage of GF eqs is their efficiency and ability to incorporate collisionless damping**
  - Also relatively easy to work with and simplify in various limits
  - Generally well behaved numerically, conservative
  - Easier than GK to interpret results
  - Right compromise between accuracy and efficiency?
- **Weakness relative to direct GK is additional simplifications**
  - No nonlinear Landau damping, drift res not exact
  - One model is GF/GK working in tandem for efficiency
- **Weakness relative to Braginskii-like eqs is presence of closure terms which are non-local in real space**
  - Braginskii assumes very high collisionality (and can be poorly behaved at low collisionality). GF assumes GK ordering is valid.

- **Extra slides**

Some generalization required for MHD-like problems

- **Need to avoid simplified equilibrium, keep full B perturbation and kink term**

The gyrocenter velocity is then given by

$$\dot{\mathbf{X}} = v_{\parallel}(\hat{\mathbf{b}} + \frac{\langle \delta \mathbf{B}_{\perp} \rangle}{B}) + \mathbf{v}_E + \mathbf{v}_d, \tag{3.3}$$

where the angular brackets denote gyroangle averages. The first term on the right represents free streaming along the total magnetic field. The second term is the gyroaveraged $\mathbf{E} \times \mathbf{B}$ drift velocity, $\mathbf{v}_E = \frac{c}{B}\hat{\mathbf{b}} \times \nabla \langle \phi \rangle$. $\mathbf{v}_d$ is the combined curvature and ∇B drift velocity. In general, $\mathbf{v}_d$ can be written

$$\begin{aligned} \mathbf{v}_d &= \frac{v_{\parallel}^2}{\Omega}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) + \frac{\mu}{\Omega}\hat{\mathbf{b}} \times \nabla B \qquad (3.4) \\ &= \frac{v_{\parallel}^2 + \mu B}{\Omega B^2}\mathbf{B} \times \nabla B + \frac{v_{\parallel}^2}{\Omega B^2}\hat{\mathbf{b}} \times (\nabla \times \mathbf{B} \times \mathbf{B}). \end{aligned}$$

Using the equilibrium relations $\nabla p = \frac{1}{c}\mathbf{J} \times \mathbf{B}$ and $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$, this can be written

$$\mathbf{v}_d = \frac{v_{\parallel}^2 + \mu B}{\Omega B^2}\mathbf{B} \times \nabla B + \frac{v_{\parallel}^2}{\Omega B^2}\hat{\mathbf{b}} \times \nabla p. \tag{3.5}$$

The second term on the right is small for $\beta \ll 1$,[3] and is neglected here for simplicity and to maintain consistency with neglecting $\delta B_{\parallel}$.[4] The definition

$$\mathbf{v}_d \doteq \frac{v_{\parallel}^2 + \mu B}{\Omega B^2}\mathbf{B} \times \nabla B \tag{3.6}$$

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